SUBSURFACE ELECTROMAGNETIC PARAMETERS IN TERMS OF THE DISTRIBUTION OF CURRENT

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Electric and magnetic fields below the horizontal surface of a conducting structure have been derived in terms of the subsurface currents. The formulae obtained are valid for any subsurface electromagnetic problems, e.g. for sea-bottom magnetotellurics or mining electromagnetics. The simplifications of these general three-dimensional formulae to one-dimensional and to the basic two-dimensional situations yield a clear physical meaning to the depth-dependence of the impedance and consequently to the apparent resistivity and the phase: in all instances the impedance at a given depth and location is entirely determined by the complex mean depth of the currents flowing beneath the actual measuring point. These results are extensions of similar ones published recently for electromagnetic parameters at the surface.

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1. Introduction

All electromagnetic sounding methods relying on natural or artificially-generated fields try to determine the subsurface conductivity structures by means of some interpretational parameters (e.g. impedance,

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apparent resistivity, phase), which are derived from the distribution and/or period-dependence of the field components at the surface.

Even after the basic geo-electromagnetic induction studies in the frequency domain [e.g. Cagniard 1953, Weidelt 1972, Price 1973] and other papers dealing with different interpretational parameters (e.g. Fischer 1985 on the 2-D magnetotelluric phase; Spies and Eggers 1986 on the apparent resistivity, etc.) the physical meaning of the geophysical interpretational parameters in the frequency domain has not always become clear.

In the paper by Szarka and Fischer [1989] the surface impedance and some corresponding parameters are interpreted in a novel way, based on the distribution of the subsurface currents. The formulae derived provide clear physical meanings for the impedance, the apparent resistivity and the phase. For example, the imaginary part of the surface impedance derives directly from the depth to the centre of gravity of the currents which are in phase with the surface magnetic field; the real part, on the other hand, is determined by the mean depth of the out-of-phase currents. The apparent resistivity — depending on how it is defined — is always related to the period-dependence of some function of the real and/or imaginary part of the complex depth of the current system; the phase tangent, in turn, is the ratio of the mean depths of in-phase and out-of-phase currents.

In this paper the subsurface impedance formulae will be given in analogous fashion to the expressions derived for the surface values by Szarka and Fischer [1989]. Similarly to this latter paper, all fields are taken to be functions of $x$, $y$, and $z$, and $\mu$ and $\varepsilon$ are assumed constant, independent of space and time. Maxwell’s equations, in which displacement currents are neglected, can then be written:

$$\oint H \, dl = \iint j \, dS, \quad (1)$$

$$\oint E \, dl = -i\omega \mu \iint H \, dS, \quad (2)$$

$$\oint H \, dl = 0, \quad (3)$$

$$\oint E \, dl = Q. \quad (4)$$

For the horizontal electromagnetic components the resulting equations derived by Szarka and Fischer [1989] are the following
\[ H_x(x,y,0) = -\int_0^\infty \left[ j_y + \frac{\partial H_z}{\partial x} \right] dz, \quad (5) \]

\[ H_y(x,y,0) = \int_0^\infty \left[ j_x - \frac{\partial H_z}{\partial y} \right] dz, \quad (6) \]

\[ E_z(x,y,0) = i\omega \mu \int_0^\infty \left[ j_x - \frac{\partial H_z}{\partial y} \right] dz - \int_0^\infty \frac{\partial E_x}{\partial x} dz, \quad (7) \]

\[ E_y(x,y,0) = i\omega \mu \int_0^\infty \left[ j_y + \frac{\partial H_z}{\partial x} \right] dz - \int_0^\infty \frac{\partial E_z}{\partial y} dz. \quad (8) \]

It will be shown that very similar equations can be derived for the subsurface electromagnetic components.

2. Depth-dependence of subsurface electromagnetic components

In Fig. 1 a closed line-integral is shown for a domain extending from an arbitrary depth \( z \) to infinity. The horizontal electric and magnetic fields can be obtained by integration over the domain from \( z \) to infinity.

\[ H_x(x,y,z) = -\int_z^\infty \left[ j_y(x,y,z) + \frac{\partial H_z(x,y,z)}{\partial x} \right] dz', \quad (9) \]

\[ H_y(x,y,z) = \int_z^\infty \left[ j_x(x,y,z) - \frac{\partial H_z(x,y,z)}{\partial y} \right] dz', \quad (10) \]

\[ E_z(x,y,z) = i\omega \mu \int_z^\infty H_x(x,y,z') dz' - \int_z^\infty \frac{\partial E_x(x,y,z')}{\partial x} dz', \quad (11) \]

\[ E_y(x,y,z) = i\omega \mu \int_z^\infty -H_y(x,y,z') dz' - \int_z^\infty \frac{\partial E_z(x,y,z')}{\partial y} dz'. \quad (12) \]
The form given to eqs (7) and (8) for the surface electric fields derives from the use of partial integrations, as shown below for the case of the $E_z$ component:

$$
\int_0^z H_z(x,y,z) \, dz = \left[ z \, H_y \right]_0^z - \int_0^z z \, \frac{\partial H_y}{\partial z} \, dz = \int_0^z \frac{\partial H_z}{\partial z} \, dz = \int_0^z \left( j_x - \frac{\partial H_y}{\partial y} \right) \, dz
$$

(13)
However, if we integrate from \( z = 0 \) to infinity the term \( zH_z \) does not disappear, and we obtain

\[
\int_{z}^{\infty} H_x(x, y, z') \, dz' = [z \ H_z]_0^\infty - \int_{z}^{\infty} \frac{\partial H_x(x, y, z')}{\partial z'} \, dz' =
\]

\[
= -z \ H_x(x, y, z) - \int_{z}^{\infty} \frac{\partial H_x(x, y, z')}{\partial z'} \, dz' = \]

\[
= \int_{z}^{\infty} \left[ j_z(x, y, z') - \frac{\partial H_z(x, y, z')}{\partial y'} \right] \, dz' - z \ H_x(x, y, z) =
\]

\[
= \int_{z}^{\infty} (z' - z) \left[ j_z(x, y, z') - \frac{\partial H_z(x, y, z')}{\partial y'} \right] \, dz'. \tag{14}
\]

On the basis of eq. (14) the following expressions can therefore be derived for the horizontal electric components:

\[
E_x(x, y, z) = i \omega \mu \int_{z}^{\infty} (z' - z) \left[ j_x(x, y, z') - \frac{\partial H_z(x, y, z')}{\partial y} \right] \, dz' - \int_{z}^{\infty} \frac{\partial E_x(x, y, z')}{\partial x} \, dz' \tag{15}
\]

\[
E_y(x, y, z) = i \omega \mu \int_{z}^{\infty} (z' - z) \left[ j_y(x, y, z') + \frac{\partial H_z(x, y, z')}{\partial x} \right] \, dz' - \int_{z}^{\infty} \frac{\partial E_y(x, y, z')}{\partial y} \, dz' \tag{16}
\]

We would mention here the similarity between expressions (5–8), valid at the top surface, and eqs (9, 10) and (15, 16) which apply at depth. This means that here too, the magnetic field at depth is given in large part by the depth integral of the current density beneath the measuring point, while the subsurface electric field strongly depends on the first moment of this current density.

The perfectly analogous structures of the two sets of equations (5–8) for the surface fields and (9, 10, 15, 16) for the fields at an interior point, clearly demonstrate that these equations are not restricted to uniform primary inducing fields. This can best be seen if we first consider a system of uniform primary inducing fields at the upper surface of a conductor bound by a plane of surface \( z = 0 \). Then, for a given period \( T \), each of the four equations, eqs (5–8), can be factored into amplitude and structural factors. The structural factor would depend only on the conductive struc-
ture and on the period, and would therefore entirely determine the impedance tensor \( Z \) at the surface. Equations (9, 10, 15, 16), on the other hand, cannot similarly be factored because the fields at an interior point \( z > 0 \) depend on the structure above that point. For example, if we imagine a structure stable below a certain depth \( z_0 \), covered with an overburden at \( z < z_0 \) which we assume arbitrarily variable, then the fields at the \( z = z_0 \) surface will depend on this superior structure. Unless the structure for \( z < z_0 \) is 1-D and the primary fields are uniform, the primary field at \( z = z_0 \) will not be uniform. The structural features above a given depth \( z_0 \) can therefore be looked upon as producing non-uniform primary fields at the \( z_0 \) level. Since the \( z_0 \) level is arbitrary, it follows that eqs (9, 10, 15, 16) remain valid at the surface as well as inside any conducting structure, including situations where the surface of the conductor is not bound by a plane surface and where the primary inducing field is not uniform. Formulae (9, 10, 15, 16) are therefore of quite general validity; in particular they remain unchanged if the surface of the conductor exhibits complicated topography and if the primary inducing field is not uniform, as for example in the immediate vicinity of an antenna.

3. 1-D and 2-D formulae at arbitrary depths

In one-dimensional (1-D) geometry the following relations can be derived from eqs. (9, 10, 15, 16):

(a) when current flow is along the \( x \) axis:

\[
E_x(z) = i\omega \mu \int_{-\infty}^{\infty} (z' - z) j_y(z') \, dz' \tag{17}
\]

\[
H_y(z) = \int_{-\infty}^{\infty} j_z(z') \, dz' \tag{18}
\]

(b) when current flow is along the \( y \) axis:

\[
E_y(z) = i\omega \mu \int_{-\infty}^{\infty} (z' - z) j_y(z') \, dz' \tag{19}
\]

\[
H_z(z) = -\int_{-\infty}^{\infty} j_z(z') \, dz' \tag{20}
\]
With two-dimensional (2-D) structures we must distinguish between \(E\)- and \(H\)-polarization configurations and we take the strike to be in the direction of the \(x\) axis.

(a) In \(E\)-polarization:

\[
E_x(y,z) = i \omega \mu \int \left[ j_x(y,z') - \frac{\partial H_y(y,z')}{\partial y} \right] dz'
\]  
(21)

\[
H_y(y,z) = \int \left[ j_y(y,z') - \frac{\partial H_x(y,z')}{\partial y} \right] dz'
\]  
(22)

(b) In \(H\)-polarization:

\[
E_y(y,z) = i \omega \mu \int \left( z - z' \right) j_x(y,z') \, dz' - \int \frac{\partial E_x(y,z')}{\partial y} \, dz'.
\]  
(23)

\[
H_x(y,z) = - \int j_y(y,z') \, dz'
\]  
(24)

4. Impedance formulae at arbitrary depth

The impedance tensor is given in terms of ratios of horizontal electric and magnetic components. In the one-dimensional geometry the tensor component \(Z_{xy}(z)\) takes the simple form:

\[
Z_{xy}(z) = -i \omega \mu \left[ \int_z^{z'} j_x(z') \, dz' \right] = -i \omega \mu (z_1 - z) - \int_z j_x(z') \, dz'
\]  
(25)

Ignoring the factor \(-i \omega \mu\) we see that the 1-D impedance \(Z_{xy}\) is given in terms of quantities with the dimensions of depths: the 1-D impedance is simply proportional to the complex mean depth below the observation point of the currents flowing beneath depth \(z\). We saw earlier that the amplitude of this current distribution is, naturally, determined by currents flowing above the measuring point, but this does not appear explicitly in eq. (25).
The well-known constancy of the impedance in the lowest layer follows directly from eq. (25) since in this lowest, or $n$-th layer, where the current density at the top is $j_{no}$ and $k_n = -i\omega \sigma_n$, we find

\[
Z_{\alpha} (z) = -i\omega m \int_{-\infty}^{z} e^{-k_n z'} dz' + i\omega m = \int_{z}^{\infty} j_{no} e^{-k_n z'} dz' = -i\omega m (z + 1/k_n) + i\omega m = \frac{i\omega m}{k_n}
\]  

(26)

Obviously, according to eq. (26), $Z_{\alpha} (z)$ in the bottom layer does not depend on depth.

In 2-D $E$-polarization we find

\[
Z_{xy} (x,z) = -i\omega m \left[ \int_{z}^{\infty} \left[ j_x (y,z') - \frac{\partial H_y (y,z')}{\partial y} \right] dz' \right] - z = \frac{-i\omega m (z_e - z)}{k_n}
\]

(27)

and in $H$-polarization

\[
Z_{xy} (y,z) = -i\omega m \left[ \int_{z}^{\infty} j_y (y,z') dz' \right] - z + \frac{\int_{z}^{\infty} \frac{\partial E_z (y,z')}{\partial y} dz'}{\int_{z}^{\infty} j_y (y,z') dz'} = -i\omega m (z_e - z) + f(E_o)
\]

(28)
Finally, in the general three-dimensional situation the component $Z_{xy}(x,y,z)$ can be written as follows:

$$Z_{xy}(x,y,z) = -\frac{E_x}{H_y} - i\omega \mu \int z' \left[ j_z(x,y,z') - \frac{\partial H_x(x,y,z')}{\partial y} \right] dz' + \int z' \left[ j_z(x,y,z') - \frac{\partial H_z(x,y,z')}{\partial y} \right] dz'$$

$$+ \int z' \frac{\partial E_z(x,y,z')}{\partial x} dz' + i\omega \mu z - i\omega \mu (z_{3-D} - z) + f_{3-D} (E_z)$$

(29)

where

$$z_{3-D} = \frac{\int z' \left[ j_z(x,y,z') - \frac{\partial H_x(x,y,z')}{\partial y} \right] dz'}{\int z' \left[ j_z(x,y,z') - \frac{\partial H_z(x,y,z')}{\partial y} \right] dz'}$$

and

$$f_{3-D} (E_z) = \frac{\int z' \frac{\partial E_z(x,y,z')}{\partial x} dz'}{\int z' \left[ j_z(x,y,z') - \frac{\partial H_z(x,y,z')}{\partial y} \right] dz'}$$

Equation (29) is clearly more complicated than eqs. (25-28), but on the basis of its one- and two-dimensional limiting behaviour all three-dimensional features can easily be understood. The other impedance components can similarly be derived.
5. New way of looking at an old example

In Fig. 2 we show the magnetotelluric phase distribution inside the upper layer of a two-layered half-space, with $\sigma_2/\sigma_1 = 100$. For the two logarithmic axes we have, respectively, the relative depth $z/h$ (where $h$ is the thickness of the first layer) and a normalized wavelength $\lambda_1/h = \sqrt{T}$.

![Graph showing magnetotelluric phase distribution](image)

*Fig. 2. Magnetotelluric phase $\Phi$ inside the first layer of a two-layer half-space $(\sigma_1, \sigma_2, h)$, when $\sigma_2/\sigma_1 = 100$, as a function of the relative wavelength $\lambda_1/h$ in the upper layer and for various relative depths $z/h$ inside this layer. In addition to the five phase curves for $z/h = 0.9, 0.99, 0.999$ and $0.9999$, several straight lines connecting points of constant phase are shown.*

2. ábra. Magnetotellurikus fázis $(\Phi)$ kétéletes felület első rétegében $(\sigma_1, \sigma_2, h)$, $\sigma_2/\sigma_1 = 100$ esetén, a felső rétegbeli relatív hullámszám $(\lambda_1/h)$ függvényében, különböző relatív mélységekre $(z/h)$. A $z/h = 0, 0.9, 0.99, 0.999$ és 0.9999 értékekre kiszámolt öt fázisdiagramon kívül az állandó fázisokat összekötő egyenes vonalakat is ábrázoltuk.

Рис. 2. Магнитотеллурическая фаза $(\Phi)$ в первом слое двухслойного полупространства $(\sigma_1, \sigma_2, h)$ для случая $\sigma_2/\sigma_1 = 100$ как функция относительного волнового числа $(\lambda_1/h)$ в верхнем слое, при различных относительных глубинах $(z/h)$. Помимо пяти фазовых диаграмм, рассчитанных для $z/h = 0, 0.9, 0.99, 0.999$ и $0.9999$ показаны также и прямые, соединяющие постоянные фазы.
(where $\lambda_1$ is the wavelength in the first layer, $\lambda_1 = 2\pi s$, where $s$ is the skindepth).

The phase plots shown are not new but nevertheless they seem to be somewhat surprising: at depths which are increasing by exponentially decreasing amounts we find the same phase behaviour as on the top surface, but at smaller and smaller $\lambda_1/h \sim \sqrt{T}$ values. Expressed mathematically we find identical phase values at all those subsurface sites for which the normalized local wavelength $\lambda_1/(h-z)$ is the same. In other words, the geometrical set of subsurface observation sites having identical phases is given by the equation:

$$\lambda_1(0)/h = \lambda_1(z)/(h-z) \quad (30)$$

As a practical consequence of eq. (30) we find that any surface phase value measured at period $T(0)$ will be obtained at depth $z$ at another period $T(z)$, which (since $\lambda_1$ is proportional to $\sqrt{T}$) is given by

$$T(z) = T(0) \left(1 - z/h \right)^2 \quad (31)$$

This relation is valid only inside the first layer, i.e. for $0 \leq z \leq h$. According to eq. (31) a surface field parameter measured at a period of, say 1000 s can be obtained at a depth $z=0.9 \, h$ at 10 s. Furthermore what happens at the surface in the period range 10–1000 s occurs in the 0.1–10 s range at the depth $z=0.9 \, h$.

The self-repetition of the phase anomaly in Fig. 2 means that

(a) the shape of the phase anomaly is determined exclusively by the resistivity contrast and

(b) the shift of the phase anomaly along the $\lambda_1/h$ axis is determined by the thickness of layer 1 between the measuring site and the layer boundary.

Points (a) and (b) clearly indicate that the phase anomaly at any depth can be described entirely by what happens below the measuring site. In other words: the phase anomaly at any depth is completely determined by the currents flowing below the observation site.

Of course, the phase anomaly at depth can also be derived starting from the surface anomaly and taking into account the current distribution between the surface and the observation site. In this section we have pointed
out the importance of the first, alternative approach in a simple two-layered case.

We believe that a careful consideration of current and charge distribution below the observation site any 2-D or 3-D problem could help at least in the intuitive understanding of the problem. Such an approach might have direct bearings on sea-bottom magnetotellurics [e.g. Filloux 1977] or in mining electromagnetics [e.g. Takacs 1989].

6. Conclusion and significance of the results

The above equations do not give direct solutions for any particular 3-D, 2-D or 1-D problem, but give closer insight into the behaviour of the impedance tensor calculated from horizontal electric and magnetic components at an arbitrary depth. The impedance in any subsurface electromagnetic problem is directly related to the complex mean depth of the currents flowing below the observation point.

The equations derived are simple extensions of the results of Szarka and Fischer [1989] and can be obtained directly by replacing the lower integration limit of zero by the arbitrary limit z>0 in the original formulae. In other words, the apparent resistivity and the phase are given by the same expressions for points inside or at the surface of a conducting structure.

As we have shown this straightforward extension implies that the formulae we have derived are of very general validity: they remain true even when the surfaces of the conductor exhibits a complicated topography and when the primary inducing fields are not uniform.

The formulae derived in this paper are valid for any subsurface electromagnetic problem, e.g., for sea-bottom magnetotellurics or mining electromagnetics, where interpretational problems have sometimes been especially difficult. The simple connection they establish between the commonly-used impedance parameters at depth and the current distribution beneath this depth should render them useful in applications of subsurface electromagnetic exploration methods.
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FELSZÍNALATTI ELEKTROMÁGNESES PARAMÉTEREK A MÉLYBELI
ÁRAMELOSZLÁS FÜGGVÉNYÉBEN

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Az inhomogén vezető közegek horizontális felszíne alatti elektromos és mágneses teret
a mélybeli árameloszlás függvényében tárgyaljuk. A levezetett formulák bármely felszín-
alatti elektromágneses problémára (pl. tengerfenék-magnetotellurikára vagy bányászati
geofizikára) alkalmazhatók. Az általános háromdimenziós összefüggéseknek egydimen-
ziós és az alapvető kétdimenziós esetekre történő leegyszerűsítése az impedancia mély-
ségfüggésére, következképpen a láthatólagos fajlagos elterülésre és a fázisra világos
fizikai értelmezést nyújt: bármely tetszőleges felszínalatti pontban az impedanciát az illető
pont alatti áramok komplex átlagmélysége határozza meg. Az eredmények a felszín
elektromágneses paraméterekre vonatkozó korábbi eredmények kiterjesztésének tekint-
etnek.
ПОДПОВЕРХНОСТНЫЕ ЭЛЕКТРОМАГНИТНЫЕ ПАРАМЕТРЫ КАК ФУНКЦИЯ ГЛУБИНОГО РАСПРЕДЕЛЕНИЯ ТОКА

Л. САРКА, Г. ФИШЕР

Электрическое и магнитное поле под горизонтальной поверхностью неоднородной проводящей среды рассматривается как функция глубинного распределения тока. Выведенные формулы могут применяться в решении любых подповерхностных электромагнитных проблем, например, в магнитотеллурике морского дна или в подземной геофизике.

Упрощенное сведение общего трехмерного случая к одномерному и основным двумерным дает ясную физическую интерпретацию зависимости импенсии от глубины и, следовательно, кажущихся удельных сопротивлений и фаз импенсии в произвольной точке под некоторой поверхностью определяется комплексной средней глубиной токов ниже данной точки.

Результаты могут рассматриваться в качестве расширения выводов, полученных ранее по поверхностным электромагнитным параметрам.