COMMENT ON 'ASPECTS OF CHARGE-ACCUMULATION IN D.C. RESISTIVITY EXPERIMENTS' BY Y. LI AND D. W. OLDENBURG

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ABSTRACT


The paper by Li and Oldenburg (1991) gives an important insight into d.c. charge accumulation problems. Nevertheless, their derivation concerning the role of the permittivity of the medium is not as straightforward as it could be. Another question, worth discussing, is the problem of double layers, which is missing from the authors' paper.

THE DERIVATION OF THE RELATION BETWEEN FREE, POLARIZATION AND TOTAL CHARGES

In the differential Maxwell equation

$$\nabla \cdot \mathbf{D} = \rho_f,$$  (1)

$\rho_f$ is the volumetric charge density due to real charge carriers, i.e. ions, electrons. This can be shown using a one-step derivation of the first Maxwell equation and (1), i.e.

$$\nabla \cdot \mathbf{j} = \nabla \cdot (\nabla \times \mathbf{H}) = \frac{\partial \mathbf{D}}{\partial t} = -\frac{\partial \rho_f}{\partial t}.$$

In conducting media these charges will 'flow', since they are 'free to move' and are called 'free charges'. Thus the volumetric charge density is denoted by $\rho_f$.

The source density of the electric field $\mathbf{E}$ can be determined from (1) as follows.

In the linear and isotropic medium assumed by Li and Oldenburg, and using the same notation as in (1), we get

$$\varepsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \varepsilon = \rho_f.$$

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Adding \( \varepsilon_0 \nabla \cdot E \) to both sides, it can be seen that both \( \rho_t \) and the permittivity \( \varepsilon \) influence the formation of charges i.e.

\[
e_0 \nabla \cdot E = \rho_t - (\varepsilon - \varepsilon_0) \nabla \cdot E - E \cdot \nabla \varepsilon.
\]

Since

\[
(\varepsilon - \varepsilon_0) \nabla \cdot E + E \cdot \nabla \varepsilon = \nabla \cdot \left( (\varepsilon - \varepsilon_0) E \right),
\]

and the polarization vector \( \mathbf{P} \) is defined as

\[
\mathbf{P} = (\varepsilon - \varepsilon_0) \mathbf{E},
\]

we get

\[
e_0 \nabla \cdot E = \rho_t - \nabla \cdot \mathbf{P},
\]

and since

\[
- \nabla \cdot \mathbf{P} = \rho_p,
\]

where \( \rho_p \) is the polarization charge density, we get

\[
e_0 \nabla \cdot E = \rho_t + \rho_p. \tag{3}
\]

In source-free regions,

\[
\nabla \cdot \mathbf{j} = 0. \tag{4}
\]

From (4) and according to Ohm’s law

\[
\sigma \nabla \cdot E + E \cdot \nabla \sigma = 0.
\]

Rearranging and multiplying both sides by \( \varepsilon_0 \), we get

\[
e_0 \nabla \cdot E = - \varepsilon_0 \mathbf{E} \cdot \frac{\nabla \sigma}{\sigma}. \tag{5}
\]

Using the definition of the ‘total’ charge density

\[
\rho_t = \rho_t + \rho_p, \tag{6}
\]

we get the final relation from (3) and (5),

\[
\rho_t = - \varepsilon_0 \mathbf{E} \cdot \frac{\nabla \sigma}{\sigma}. \tag{7}
\]

The precise role of the permittivity, as given by Li and Oldenburg (1991), can now be given using a joint interpretation of (6) and (7): \( \rho_t \) is determined by the electrical conductivity (see (7)), but in an inhomogeneously polarizable medium, when a part of the charge is supplied by polarization charges, only the complementary part will be provided by free charges.

As an illustration, a point source near an interface between two half-spaces are presented in three different cases (see Fig. 1 after Szarka 1990):

(a) a static charge \( Q \) in medium 1, with permittivities \( \varepsilon_1 \) and \( \varepsilon_2 \);
(b) a current source \( I \) in medium 1, with conductivities \( \sigma_1 \) and \( \sigma_2 \), if \( \varepsilon_1 = \varepsilon_2 = \varepsilon_0 \);
Fig 1: Point source near to a plane interface between two adjacent half-spaces: (a) electrostatic problem with $\varepsilon_1 \neq \varepsilon_2$; (b) d.c. problem with $\sigma_1 \neq \sigma_2$ and with $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$; and (c) d.c. problem with $\sigma_1 \neq \sigma_2$ and with $\varepsilon_1 \neq \varepsilon_2$.

(c) a current source $I$ in medium 1, with conductivities $\sigma_1$ and $\sigma_2$, and with permittivities $\varepsilon_1$ and $\varepsilon_2$ ($\varepsilon_1 \neq \varepsilon_2$).

Using the image charge method (which is based on a mathematical substitution of the real surface charge system by virtual image point sources) for the three cases, the image sources $Q^{(1)}$, $Q^{(2)}$, $I^{(1)}$ and $I^{(2)}$ can be determined as follows:

\begin{align*}
Q^{(1)} &= Q_{k_{12}(e)}, & I^{(1)} &= I_{k_{12}(\sigma)}, \\
Q^{(2)} &= Q_{t_{12}(e)}, & I^{(2)} &= I_{t_{12}(\sigma)},
\end{align*}

where $k_{12}(e)$ and $k_{12}(\sigma)$ are the reflection coefficients and $t_{12}(e)$ and $t_{12}(\sigma)$ are the transmission coefficients. They are defined as follows:

\begin{align*}
k_{12}(e) &= \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, & k_{12}(\sigma) &= \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}, \\
t_{12}(e) &= \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2}, & t_{12}(\sigma) &= \frac{2\sigma_2}{\sigma_1 + \sigma_2}.
\end{align*}
Table 1. Polarization, free and total charge systems due to a point source at a
distance \( d \) from a plane interface between two homogeneous half-spaces.

<table>
<thead>
<tr>
<th></th>
<th>(a) ( Q, \varepsilon_1, \varepsilon_2 )</th>
<th>(b) ( I, \sigma_1, \sigma_2, \varepsilon_0 )</th>
<th>(c) ( I, \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_s(R) )</td>
<td>( \frac{\varepsilon_0 Qd k_{12}(\varepsilon)}{2\pi \varepsilon_1 R^3} )</td>
<td>0</td>
<td>( \frac{I d k_{12}(\sigma_1) - 1}{4\pi R^3} ) ( \left( \frac{\varepsilon_1 - \varepsilon_0}{\sigma_1} \right) )</td>
</tr>
<tr>
<td>( \tau_f(R) )</td>
<td>0</td>
<td>( \frac{\varepsilon_0 I d k_{12}(\varepsilon)}{2\pi \varepsilon_1 R^3} )</td>
<td>( \frac{I d k_{12}(\sigma_1) - 1}{4\pi R^3} ) ( \left( \frac{\varepsilon_2 - \varepsilon_1}{\sigma_2} \right) )</td>
</tr>
<tr>
<td>( \tau_t(R) )</td>
<td>( \frac{\varepsilon_0 Qd k_{12}(\varepsilon)}{2\pi \varepsilon_1 R^3} )</td>
<td>( \frac{\varepsilon_0 I d k_{12}(\sigma_1)}{2\pi \varepsilon_1 R^3} )</td>
<td>( \frac{\varepsilon_0 I d k_{12}(\sigma_1)}{2\pi \varepsilon_1 R^3} )</td>
</tr>
<tr>
<td>( \int_A \tau_t dA )</td>
<td>( \frac{Q k_{12}(\varepsilon)}{\sigma_1} )</td>
<td>( \frac{\varepsilon_0 I k_{12}(\sigma_1)}{\sigma_1} )</td>
<td>( \frac{\varepsilon_0 I k_{12}(\sigma_1)}{\sigma_1} )</td>
</tr>
</tbody>
</table>

The surface charge versions of (1) and (2), i.e.

\[
D_{\sigma_2} - D_{\sigma_1} = \tau_f \quad \text{and} \quad P_{\sigma_1} - P_{\sigma_2} = \tau_p,
\]

(where \( \tau_f \) and \( \tau_p \) are free and polarization surface charge densities) yield all EM field components, so that the polarization charges and the free charges can be derived. For the three cases, the results for \( \tau_p, \tau_f, \tau_t \), and the surface integral of \( \tau_t \) (the total induced charge) are summarized in Table 1, clearly demonstrating the meaning of (6) and (7), i.e. the precise role of free, polarization and total charges.

**ROLE OF DOUBLE LAYERS**

Equation (24) of Li and Oldenburg (1991) with their whole-space Green function gives

\[
\phi(r) = -\frac{1}{4\pi} \int \int \frac{V^2 \phi}{|r - r'|} dV' - \frac{1}{4\pi} \int \int \left( \frac{1}{|r - r'|} \frac{\partial}{\partial n} \phi - \phi \frac{\partial}{\partial n} \frac{1}{|r - r'|} \right) dS'.
\]  

Equation (8) means that with some charges outside \( V' \), the potential of a rotation-free, divergent field can be determined only if \( V^2 \phi \) is known throughout the volume \( V' \), and \( \phi \) and \( (\partial \phi / \partial n) \) are known on the bounding surface \( S' \).

Since

\[
E = -V \phi,
\]
in Poisson's equation

\[
V^2 \phi(r) = -V \cdot E(r) = -\frac{\sigma(r)}{\varepsilon_0}.
\]

Excluding a surface \( A' \) (where \( A' \) is the interface between half-spaces (1) and (2)) from the otherwise infinitely enlarged volume \( V' \), the potential formula can be given
as a sum of volumetric total charge densities and two types of surface charges as follows:

\[
\phi(r) = \frac{1}{4\pi\varepsilon_0} \int \int \rho(r') \, dV' + \frac{1}{4\pi\varepsilon_0} \int \int \frac{\tau(r')}{|r - r'|} \, dA'
\]

\[
+ \frac{1}{4\pi\varepsilon_0} \int \int \nu(r') \frac{\partial}{\partial n} \frac{1}{|r - r'|} \, dA',
\]

where \( \tau = \varepsilon_0 \left[ (\partial \phi/\partial n)_1 - (\partial \phi/\partial n)_2 \right] \) the simple surface charge density, and \( \nu = \varepsilon_0 (\phi_1 - \phi_2) \), the moment of the surface double layer. (The direction of the normal vector \( n \) is defined as pointing from medium 1 to medium 2).

For the potential \( \phi \) between media (1) and (2), two different boundary conditions can be assumed.

1. If \( \phi_1 = \phi_2 \), then double layers cannot exist and there are only surface charges on surface \( A' \).

2. If \( (\partial \phi/\partial n)_1 = (\partial \phi/\partial n)_2 \), then surface charges cannot exist and only a double layer appears on surface \( A' \).

Great care should be taken in defining the boundary conditions. It is known that the ‘electromotive forces’ make the quantity \( (\partial \phi/\partial n) \) continuous at such an interface (e.g. when a metal electrode is immersed in an electrolyte), resulting in a characteristic jump in the potential. In these cases there are double layers at the resistivity interfaces.

If we are outside a certain region, in a mathematical sense, a double layer inside the region can always be substituted by a surface charge term and vice versa. This means that from the potential distribution alone, it cannot be said whether the anomaly was produced by simple surface charges or double layers.

**Conclusions**

Free and polarization charges (in either volumetric or surface distribution) play an important role, but the question as to which charge system, simple charges or double layers, is present, is answered uniquely by the actual boundary conditions for the potential. We should not forget that, outside the anomalous region, the type of charge system does not have any influence on the potential distribution. The anomalous charge system can also be considered in terms of simple charges and double layers.

Although the basic equations can be found in physics text books, it is very important for the geophysical community to see both the precise role of the polarization charge and the double layer problems clearly.
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REFERENCES
